Quelques exemples d'algorithmes gloutons :

leur analyse et leurs applications pour la simulation numérique de phénomènes de la mécanique.

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Outline

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Interpolation is a general and classical tool for approximation

Assume you are given a set $\varphi_1, \ldots, \varphi_n, \ldots$ of linearly independent functions, given a function f that you want to approximate, the problem is

- find a familly of scalars $\{\alpha_n^M\}_{1 \le n \le M}$ such that
- $f(\zeta_m^M) = \sum_{n=1}^M \alpha_n^M \varphi_n(\zeta_m^M)$

where the *interpolation points* ζ_m^M are suitably chosen
of course, the project is generally not limited to the approximation in the only

$$X_M = \text{Span}\{\varphi_n,\ 1 \le n \le M\}$$

Background.

General issues about interpolation

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Among the classical questions raised by interpolation

- given a set of points, does the interpolant at these points exist;
- is the interpolant unique;
- how does the interpolation process compares with other approximations (in particular orthogonal projections);
- is there an optimal selection for the interpolation points;
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These question are covered in the polynomial case, though not completely and the answers are complex and rather recent

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Definition

Let $\mathcal{X}$ be a normed linear space, X be a subset of $\mathcal{X}$ and X_n be a generic n-dimensional subspace of $\mathcal{X}$. The deviation of X from X_n is

$$E(X; X_n) = \sup_{x \in X} \inf_{y \in X_n} \|x - y\|_{\mathcal{X}}.$$

The *Kolmogorov n-width* of X in X is given by

$$\begin{aligned} d_n(X, \mathcal{X}) &= \inf\{E(X; X_n) : X_n \text{ an } n\text{-dimensional subspace of X}\} \\ &= \inf_{X_n} \sup_{x \in X} \inf_{y \in X_n} \|x - y\|_{\mathcal{X}} \ . \end{aligned} \tag{1}$$

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Approximation in a space of small n-width.

Interpolation

We are looking for a constructive way of approximating in $\mathcal{X}$, we assume that $\mathcal{X} \subset \mathcal{C}^0$

We propose a greedy approach both for constructing the interpolation points and the discrete spaces X_M,

our method is hierarchical

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definition of the magic points

The first interpolating function is

$$\varphi_1 = \arg \max_{\Phi \in \mathcal{S}_1[X]} \|\Phi\|_{L^\infty(\Omega)}$$

The first interpolation point is

$$\zeta_1 = \arg \max_{x \in \Omega} |\varphi_1|$$

and we set $q_1 = \varphi_1(\cdot)/\varphi_1(\zeta_1)$ The second interpolating function is

$$\varphi_2 = \arg \max_{\Phi \in \mathcal{S}_1[X]} \|\Phi(\cdot) - \Phi(\zeta_1)q_1\|_{L^\infty(\Omega)}$$

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The recursion formula

We thus construct, by induction, the nested sets of basis functions $\{q_1, \ldots, q_M\}$ and the nested sets of interpolation points $T_M = \{\zeta_1, \ldots, \zeta_M\}, 1 \leq M \leq M_{\max},$.

For $M = 3, \ldots, M_{\max}$, we first solve the interpolation problem for $\alpha_j^{M-1}(\Phi), 1 \leq j \leq M-1$, from

$$\sum_{j=1}^{M-1} q_j(\zeta_i)\alpha_j^{M-1}(\Phi) = \Phi(\zeta_i), \quad i = 1, \ldots, M-1 \,, \tag{2}$$

and compute

$$\mathcal{I}_{M-1}[\Phi] = \sum_{j=1}^{M-1} \alpha_j^{M-1}(\Phi) q_j \,, \tag{3}$$

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We then define

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The procedure is well posed if X is of sufficently large dimension (for $M \leq M_{max} \leq \dim X$).

Note that the matrix B^M is invertible and lower triangular (the diagonal is Id).

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The Lebesgue constant

The error analysis of the interpolation procedure classically involves the Lebesgue constant $\Lambda_M = \sup_{x\in\Omega} \sum_{i=1}^{M} |h_i^M(x)|$, where the h_i^M is the associated Lagrange basis.

A (in practice very pessimistic) upper-bound for the Lebesgue constant is $2^M - 1$.

We remind also that the Lebesgue constant enters into the bound for the interpolation error as follows

Lemma

For any $u \in X$, the interpolation error satisfies

$$\|\Phi - \mathcal{I}_M\Phi\|_{L^\infty(\Omega)} \leq (1+\Lambda_M) \inf_{\psi_M \in span\{\varphi_i, 1\leq i\leq M\}} \|\Phi - \psi_M\|_{L^\infty(\Omega)}. \quad (7)$$

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$$\|\Phi - \mathcal{I}_M\Phi\|_{L^\infty(\Omega)} \leq (1+\Lambda_M) \inf_{\psi_M \in \mathit{span}\{\varphi_i, 1\leq i\leq M\}} \|\Phi - \psi_M\|_{L^\infty(\Omega)}. \quad (7)$$

We can also prove that

Theorem

Assume that there exists a sequence of finite dimensional spaces

$$X_1 \subset X_2 \subset \cdots \subset X_M \subset \cdots \subset X, \quad \dim X_M = M \tag{8}$$

such that there exists $c > 0$ *and* α *with*

$$\forall \Phi \in X, \inf_{\psi_M \in X_M} \|\Phi - \psi_M\|_X \leq c e^{-\alpha M} \tag{9}$$

then, if $\alpha > \log(4)$, *there exists* $\beta > 0$ *such that*

$$\|\Phi - \mathcal{I}_M \Phi\|_{L^\infty(\Omega)} \leq c e^{-\beta M}. \tag{10}$$

Interpolation error..... a posteriori

For an estimator on the error, let $M \leq M_{\max} - 1$, we define

$$\hat{\varepsilon}_M(\Phi) \equiv |\Phi(\zeta_{M+1}) - \mathcal{I}_M\Phi(\zeta_{M+1})|$$

Lemma : If $\Phi \in X_{M+1}$, then

$$\|\Phi(\,\cdot) - \mathcal{I}_M\Phi(\,\cdot)\|_{L^\infty(\Omega)} \leq \hat{\varepsilon}_M(\Phi)$$

Of course, in general $\Phi \notin X_{M+1}$
and hence our estimator $\hat{\varepsilon}_M(\Phi)$ is not a rigorous upper bound;

however, if $\varepsilon_M(\Phi) \to 0$ very fast,

we expect (and check) that the effectivity, $\eta_M(\Phi) \equiv \hat{\varepsilon}_M(\Phi)/\varepsilon_M(\Phi) \simeq 1$.

Furthermore, the estimator is very inexpensive – *one additional evaluation* of Φ.

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Numerical results

We consider
$\Phi(x) \equiv \Phi\left((x_1, x_2); (\mu_1, \mu_2)\right) \equiv \left((x_1 - \mu_1)^2 + (x_2 - \mu_2)^2\right)^{-1/2}$ for $x \in]0,1[^2$ and $\mu \in [-1, -0.01]^2$

M	$\varepsilon^*_{M,\max}$	$\overline{\rho}_M$	Λ_M	$\overline{\eta}_M$
8	8.30 E-02	0.68	1.76	0.17
16	4.20 E-03	0.67	2.63	0.1.
24	2.68 E-04	0.49	4.42	0.28
32	5.64 E-05	0.48	5.15	0.20
40	3.66 E-06	0.54	4.98	0.60
48	6.08 E-07	0.37	7.43	0.29

$\varepsilon^*_{M,\max}$ is the best fit error, $\overline{\rho}_M$ is the averaged ratio $\dfrac{\varepsilon}{\varepsilon^*(1+\Lambda)}$, Λ_M is the "Lebesgue" constant and $\overline{\eta}_M$ is the averaged effectivity index $\dfrac{\hat{\varepsilon}}{\varepsilon}$

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Note that we have here approximated the full set of $\Phi((x); \mu)$ with a few of them $\Phi((x); (\mu)) \simeq \sum_{i=1}^{M} \alpha_i \Phi\left((x); \mu^i\right)$

Numerical results

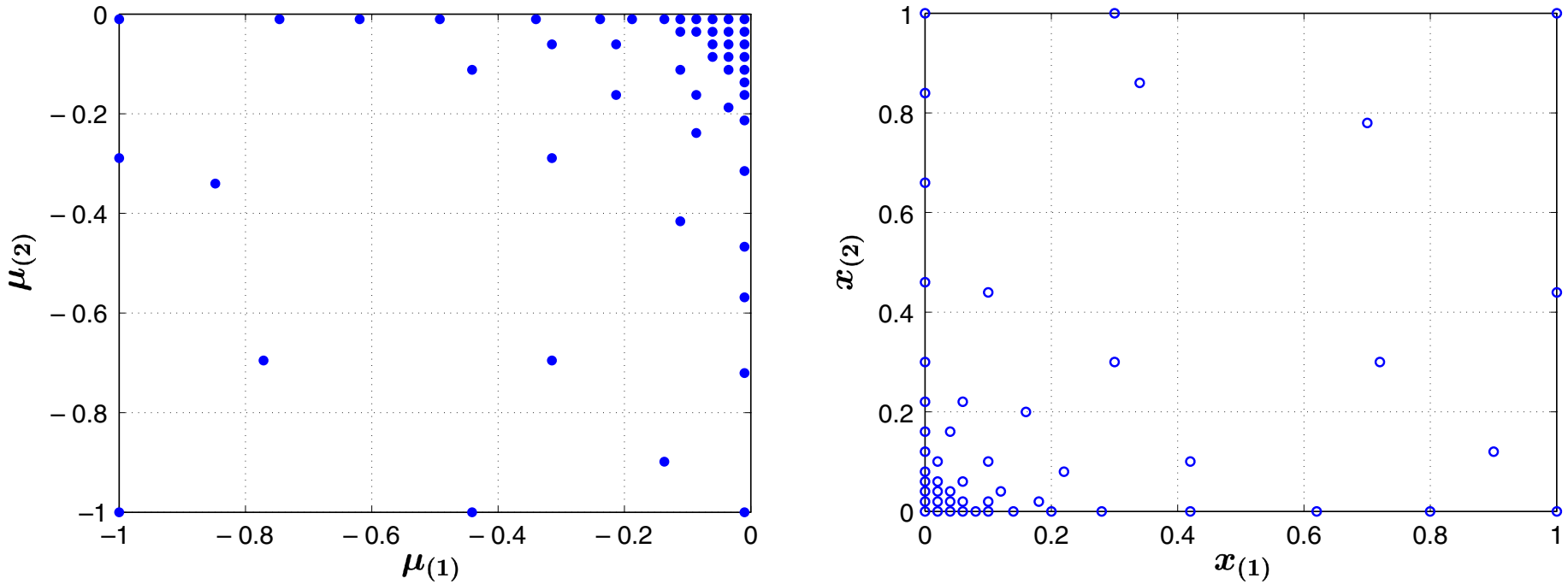

Figure: (a) Parameter sample set S_M^g, $M_{\max} = 51$, and (b) interpolation points x_m, $1 \leq m \leq M_{\max}$.

Outline

Application to polynomial interpolation

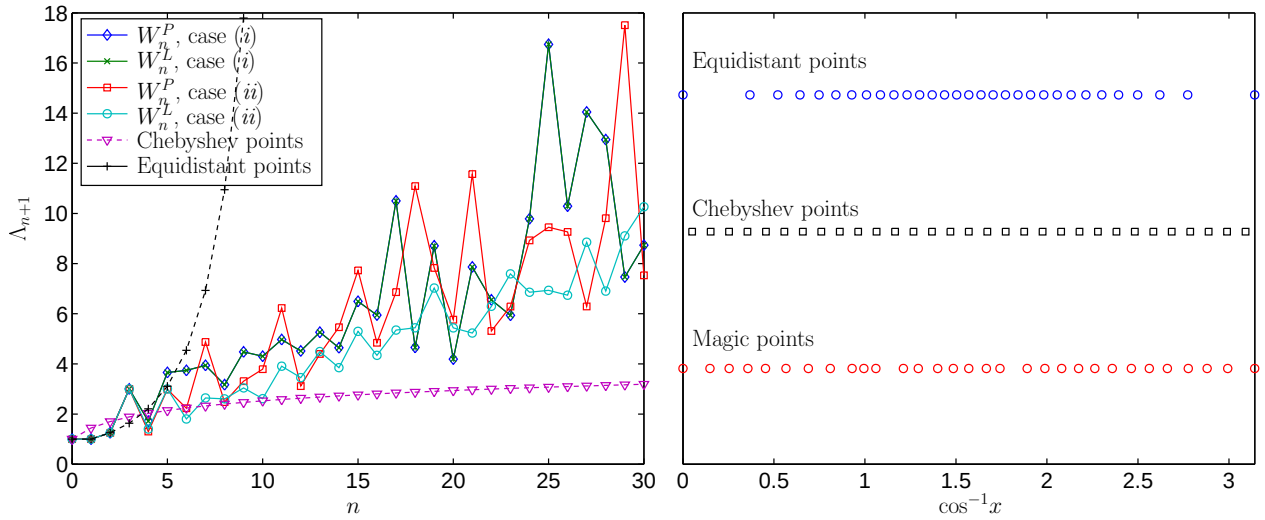

Figure: Lebesgue constant on the interval.

Application to polynomial interpolation

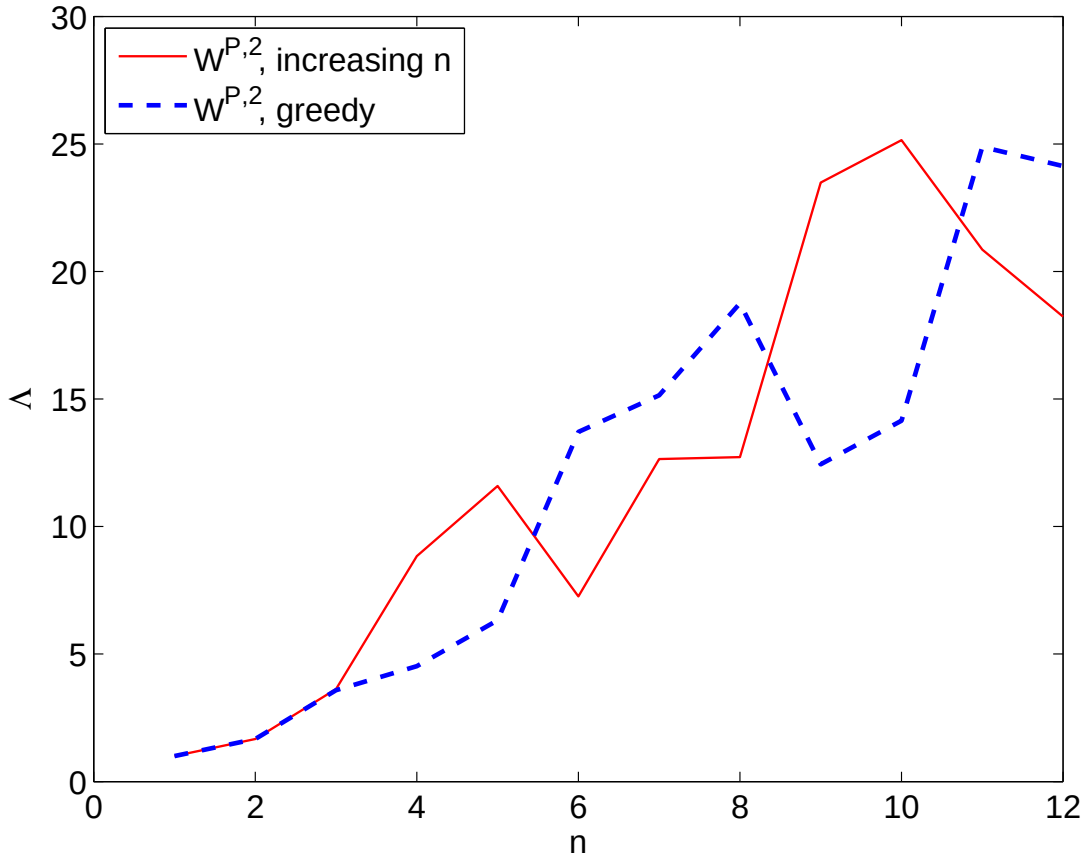

Figure: Lebesgue constant on the the triangle.

Application to polynomial interpolation

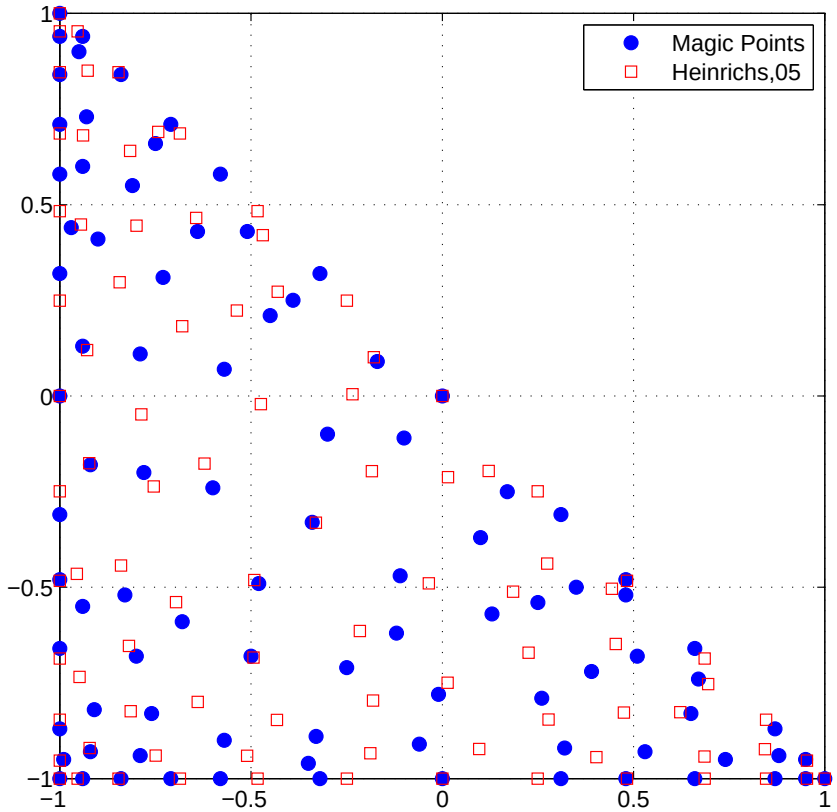

Figure: Disposition of the points on the the triangle.

Application to polynomial interpolation

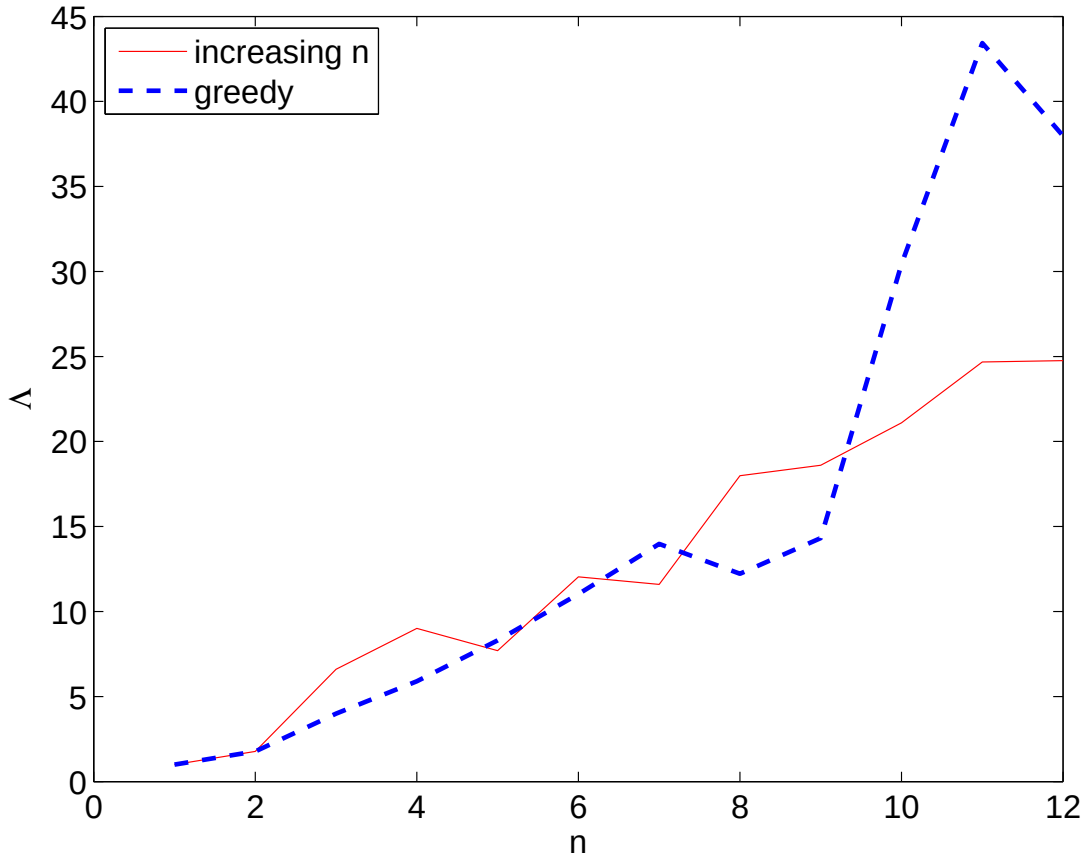

Figure: Lebesgue constant on the the pentagon.

Application to polynomial interpolation

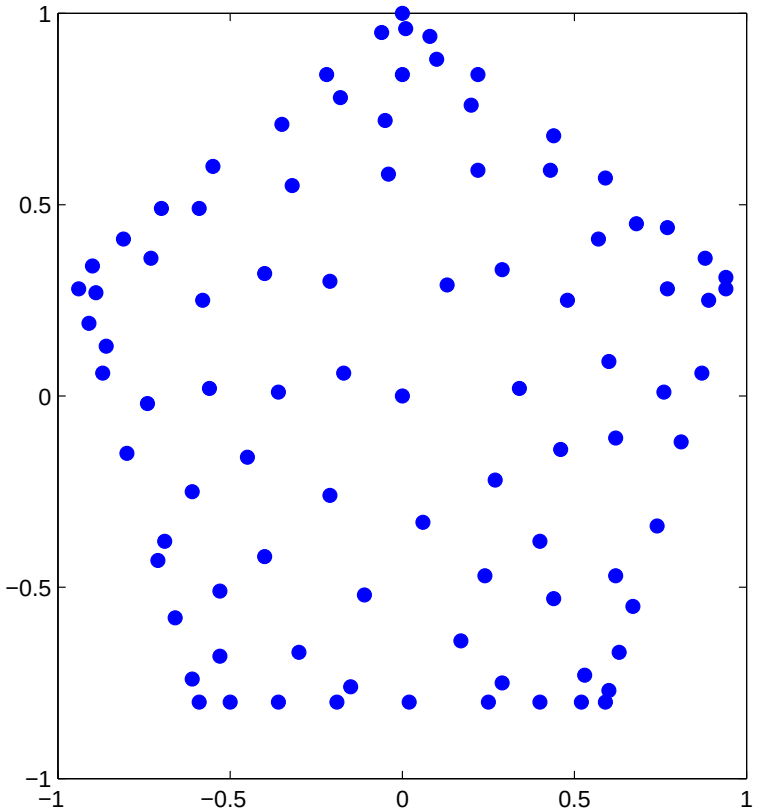

Figure: Disposition of the points on the the pentagon.

Application to polynomial interpolation

n	Magic Points	[12]	[6]
2	2.0	2.0	2.0
3	3.80	2.93	2.93
4	8.70	4.07	4.11
5	9.77	5.38	5.62
6	15.27	7.53	7.36
7	31.04	10.17	9.37
8	34.31	14.63	12.31
9	62.99	20.46	15.69

Figure: Lebesgue constant on the the tetrahedron.

Application to polynomial interpolation

Application to polynomial interpolation

n	M	$\lvert\varphi_{1d}(x_{M+1}) - \mathcal{I}_M[\varphi_{1d}(x_{M+1})]\rvert$	$\lVert\varphi_{1d} - \mathcal{I}_M\varphi_{1d}\rVert_{L^\infty}$	η_M
2	3	7.27 E−2	7.79 E−2	1.07
4	5	7.47 E−3	7.52 E−3	1.01
6	7	6.18 E−4	6.70 E−4	1.08
8	9	3.84 E−5	3.84 E−5	1.00
10	11	1.69 E−6	1.72 E−6	1.02
12	13	3.08 E−8	4.02 E−8	1.30
14	15	1.65 E−9	1.65 E−9	1.00
16	17	6.33 E−11	6.73 E−11	1.06
18	19	1.39 E−12	1.39 E−12	1.00
20	21	2.50 E−14	2.51 E−14	1.00

Table: Comparison between the error estimate and the actual error, for polynomial interpolation of $\varphi_{1d} = e^{-x^2}$.

Application to polynomial interpolation

n	M	$\lvert\varphi_{2d}(x_{M+1}) - \mathcal{I}_M[\varphi_{2d}(x_{M+1})]\rvert$	$\lVert\varphi_{2d} - \mathcal{I}_M\varphi_{2d}\rVert_{L^\infty}$	η_M
2	9	1.13 E−1	6.32 E−1	5.59
4	25	1.43 E−1	1.66 E−1	1.16
6	49	2.03 E−2	2.24 E−2	1.10
8	81	7.23 E−4	1.46 E−3	2.02
10	121	5.36 E−5	1.06 E−4	1.98
12	169	2.76 E−6	2.78 E−6	1.01
14	225	1.04 E−8	1.31 E−7	12.60
16	289	2.67 E−9	4.88 E−9	1.83
18	361	4.98 E−11	1.16 E−10	2.33
20	441	2.57 E−12	2.78 E−12	1.08

Table: Comparison between the error estimate and the actual error, for polynomial interpolation of $\varphi_{2d} = e^{-(x^2+y^2)}$.

Application to polynomial interpolation

n	M	$\lvert\varphi_{\mathrm{irr}}(x_{M+1}) - \mathcal{I}_M[\varphi_{\mathrm{irr}}(x_{M+1})]\rvert$	$\lVert\varphi_{\mathrm{irr}} - \mathcal{I}_M\varphi_{\mathrm{irr}}\rVert_{L^\infty}$	η_M
2	9	$7.95\,\mathrm{E}{-2}$	$1.59\,\mathrm{E}{-1}$	2.00
4	25	$3.88\,\mathrm{E}{-2}$	$1.47\,\mathrm{E}{-1}$	3.79
6	49	$2.44\,\mathrm{E}{-3}$	$1.95\,\mathrm{E}{-2}$	8.00
8	81	$4.26\,\mathrm{E}{-3}$	$2.42\,\mathrm{E}{-2}$	5.68
10	121	$1.37\,\mathrm{E}{-3}$	$3.74\,\mathrm{E}{-3}$	2.73
12	169	$3.75\,\mathrm{E}{-3}$	$5.66\,\mathrm{E}{-3}$	1.51
14	225	$2.96\,\mathrm{E}{-4}$	$5.69\,\mathrm{E}{-4}$	1.92
16	289	$5.01\,\mathrm{E}{-5}$	$5.80\,\mathrm{E}{-4}$	11.58
18	361	$1.29\,\mathrm{E}{-4}$	$3.00\,\mathrm{E}{-4}$	2.33
20	441	$3.09\,\mathrm{E}{-4}$	$5.72\,\mathrm{E}{-4}$	1.85

Table: Comparison between the error estimate and the actual error, for polynomial interpolation of $\varphi_{\mathrm{irr}} = |x^3y^3|$.

Outline

Reduced Basis Method : Framework of the approach.

Basics on approximation

- A lot of problems we have to face in numerical analysis and scientific computing: find u such that

$$\mathcal{F}(u) = 0 \tag{1}$$

can actually be written under a variational form : find $u \in \mathcal{X}$ such that

$$\mathcal{A}(u, v) =< f, v >, \quad \forall v \in \tilde{\mathcal{X}} \tag{2}$$

Where $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are some coherent Banach spaces, $\mathcal{A}$ is an appropriate continuous form, linear in v, and f is a given linear form.
For time dependent problem one may specify even : find $u, \forall t, u(t, ;) \in \mathcal{X}$ such that

$$m(\frac{\partial u}{\partial t}, v) + \mathcal{A}(u, v) =< f, v >, \quad \forall v \in \tilde{\mathcal{X}} \tag{2'}$$

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- the ellipticity or coercivity (Lax Milgram theorem when $\mathcal{X} = \tilde{\mathcal{X}}$) or
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that makes explicit conditions under which the problem is well posed : i.e. there exists a unique solution u to problem (1).

For nonlinear problems the conditions are various and more involved.

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The approximation can now proceed. Two families of finite dimensional spaces $\{\mathcal{X}_n\}_n$ and $\{\tilde{\mathcal{X}}_n\}_n$ are provided, that maintain the above mentioned coherence.

and the discrete space reads : find $u_n \in \mathcal{X}_n$ such that

$$\mathcal{A}_n(u_n, v_n) =< f_n, v_n >, \quad \forall v_n \in \tilde{\mathcal{X}}_n \qquad (2_n)$$

or again for time dependent problems : find $u_n, \forall t, u_n(t,;) \in \mathcal{X}_n$ such that

$$m_n(\frac{\partial u_n}{\partial t}, v_n) + \mathcal{A}_n(u_n, v_n) =< f_n, v_n >, \quad \forall v_n \in \tilde{\mathcal{X}}_n \qquad (2'_n)$$

most often further numerical quadratures are involved leading to slightly modified discrete problems

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All the art of the numerical analyst or the specialist of scientific computing tends to define the discrete spaces $\mathcal{X}_n$ and $\tilde{\mathcal{X}}_n$ ... and also $\mathcal{A}_n$.... in such a way that

- The discrete solutions u_n exist and are unique
- An error bound $\|u - u_N\|_{\mathcal{X}} \leq c \inf_{w_n \in \mathcal{X}_n} \|u - w_N\|_{\mathcal{X}}$ can be derived
- The best fit, $\inf_{w_n \in \mathcal{X}_n} \|u - w_N\|_{\mathcal{X}}$, goes to zero rapidly
- The effective computation of u_n is easy enough
- An a posteriori error providing the size of $\|u - u_N\|_{\mathcal{X}}$ is available
- An a posteriori indicator telling what to do to improve $\|u - u_N\|_{\mathcal{X}}$ is available

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- The best fit, $\inf_{w_n \in \mathcal{X}_n} \|u - w_N\|_{\mathcal{X}}$, goes to zero rapidly
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Reduced Basis Method : Framework of the approach.

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All the art of the numerical analyst or the specialist of scientific computing tends to define the discrete spaces $\mathcal{X}_n$ and $\tilde{\mathcal{X}}_n$ ... and also $\mathcal{A}_n$.... in such a way that

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What most approaches do

Most approaches tend to define a family of spaces $\mathcal{X}_n$ either by proposing

- multipurpose approximations providing good accuracy assuming some regularity holds this is the case of finite differences, finite element, finite volume, spectral... methods

then error in lower order spaces (Aubin's trick)

- nonlinear approximations based on a posteriori indicators that allows refinements or based on multiresolution analysis

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What reduced basis method propose

Here what we want to use is an a priori knowledge of a reduced space X much smaller than $\mathcal{X}$ where the solution to (1) should be sought

We present here two classes of problems where this strategy can be used

- parameter dependent problems
- hierarchical geometry for the domain

in both cases the space X is conceived from the use of a more standard approximation methods you do not have to forget your favorite method... it is more the opposite in a first step

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Outline

Parameter dependent problems.

Basics

Let us consider a class of problems depending on some parameters:

$$\mathcal{F}(u, \mu) = 0 \qquad (1')$$

and the parameter μ belongs to R^d (or some brick in R^d)

- This is the case for instance in a dimensional problem where some parameters have to be optimized for some purpose
- This can equally be the case for an inverse problem in parameter identification.
- The solution $u = u(\mu)$ of (1') is sought in some space $\mathcal{X}$ for any given parameter μ
- The dependancy in μ of the solution $u(\mu)$ is most often regular.

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The reduced basis space and approximation

- Define $X = \text{Span}\{u(\mu),\ \mu \in \mathcal{D}\}$ then looking for the solution in X instead of $\mathcal{X}$ (generally a Sobolev space) is already a valuable indication.....
- In order to apprehend in which sense the good behavior of X should be understood, it is helpfull to introduce the notion of n-width following Kolmogorov

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Definition

Let $\mathcal{X}$ be a normed linear space, X be a subset of $\mathcal{X}$ and X_n be a generic n-dimensional subspace of $\mathcal{X}$. The deviation of X from X_n is

$$E(X; X_n) = \sup_{x \in X} \inf_{y \in X_n} \|x - y\|_{\mathcal{X}}.$$

The *Kolmogorov n-width* of X in X is given by

$$\begin{aligned} d_n(X, \mathcal{X}) &= \inf\{E(X; X_n) : X_n \text{ an } n\text{-dimensional subspace of X}\} \\ &= \inf_{X_n} \sup_{x \in X} \inf_{y \in X_n} \|x - y\|_{\mathcal{X}} \ . \end{aligned} \tag{11}$$

The n-width of X thus measures the extent to which X may be approximated by a n-dimensional subspace of $\mathcal{X}$.

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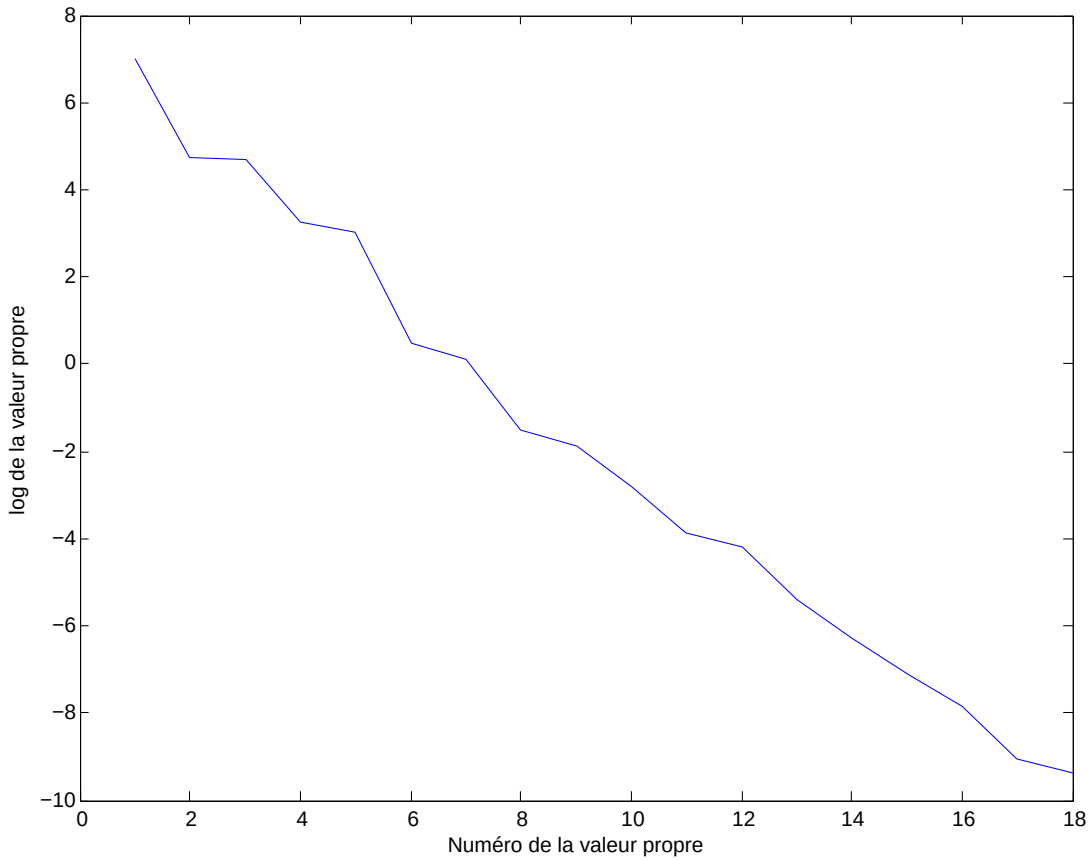
Evaluation of the n-width of the set of solutions.

- PCA in appropriate norms



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The reduced basis space and approximation

- Of course X is rarely known but $X_n = \text{Span}\{u(\mu_k),\ k = 1, ..., n\}$ where μ_k are properly chosen
- The solution to (1') for other values of μ is then approximated through a Galerkin process.
- The best fit approximation is often exponential in n and a random log repartition of the sample values μ_k is often better than other obvious choices.
  - Almroth B.O., Stern P., Brogan F.A.(1978)
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An example

Thermal fin problem

$$a(u, v; \mu) := \int_\Omega k \nabla u \nabla v + \mathrm{Bi} \int_{\partial\Omega \setminus \Gamma_1} uv = \int_{\Gamma_1} v$$

The parameters are :

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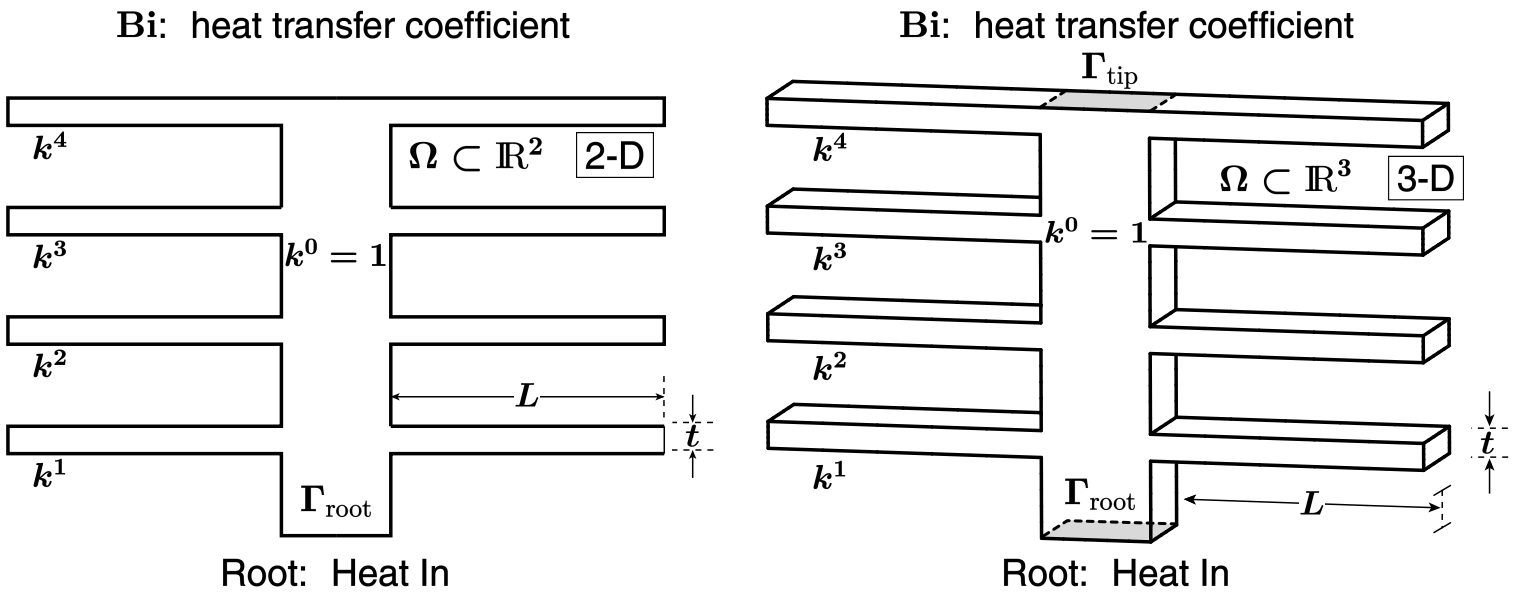

Figure: the fin geometry.

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The parameters are :

- the conductivities $k_i, i = 1, .., 4$
- the dimensions L and t
- the Biot number Bi
- The design space is $\mathcal{D} = [0.1, 10]^4 \times [0.01, 1] \times [2, 3] \times [0.1, 0.5]$

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... and its approximation

We choose randomly N points in the design space $\mathcal{D}$
and compute the solutions for these points by a finite element method (say).
The reduced basis method is then implemented, from these solutions, and the error on the averaged temperature at the foot of the fin is

N	10	20	30	40	50
Error	1.610^{-1}	1.610^{-2}	2.410^{-3}	7.210^{-4}	3.110^{-4}

Note that even for $N = 50$ there are less than 2 points per parameter direction, exponential convergence

N appears rather not much dependent on the number of parameters, actually $N(d) << N(1)^d$!!!

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Black-Box implementation

In order to solve the reduced basis problems in real time, some preliminary computations — off line — have to be made :

- e.g. the stiffness matrix $\mathcal{A}_{i,j} := a(u(\mu_i), u(\mu_j); \mu)$.

Note that $\Omega = \cup\Omega_\ell$ hence : $\int_\Omega = \sum_{\ell=0}^{6} \int_{\Omega_\ell}$

Note also that through a change of variable, each integral over the Ω_ℓ can be written as a linear combinaison of $\int_{-1}^{1}\int_{-1}^{1} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$ and $\int_{-1}^{1}\int_{-1}^{1} \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$

so that the original problem

$$\int_\Omega k\nabla u\nabla v + \mathrm{Bi}\int_{\partial\Omega\setminus\Gamma_1} uv = \int_{\Gamma_1} v$$

can be rewritten as

$$\sum_{p=1}^{P} g_p(\mu) a_p(u,v) = \ell(v)$$

- where the bilinear forms a_p are parameter independent

Black-Box implementation

In order to solve the reduced basis problems in real time, some preliminary computations — off line — have to be made :

- e.g. the stiffness matrix $\mathcal{A}_{i,j} := a(u(\mu_i), u(\mu_j); \mu)$.

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Black-Box implementation $\sum_{p=1}^{P} g_p(\mu) a_p(u, v) = \ell(v)$

The implementation is then simplified as

- offline, the different $a_p(\zeta_i, \zeta_j)$ are precomputed,
  the $\zeta_i := v(\mu_i)$ being the reduced basis elements
- online, the computation of the stiffness matrix requires only PN^2 computations,
  instead of $\mathcal{N}^2$, where $\mathcal{N}$ is the dimension of the finite element basis.

$$A_N = \sum_{p=1}^{P} \theta_p(\mu) a_p(\zeta_i, \zeta_j)$$

- online again the stiffness matrix is inverted in N^3 operations (direct inversion)

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Outline

Error Estimates

outputs

Most of the time, the complete knowledge of the solution of partial differential equation is not required.

e.g., assume you have the following problem : find $u \in X$

$$a(u, v; \mu) =< f, v >, \quad \forall v \in X$$

What is required, generally, is outputs computed from the calculated solution :

- then compute the following output $s = s(u)$

The discretization then proceeds : find $u_\delta \in X_\delta$

$$a(u_\delta, v_\delta) =< f, v_\delta >, \quad \forall v_\delta \in X_\delta$$

and the approximated output is given by $s_\delta = s(u_\delta)$

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How valid is the approximated output $s_\delta = s(u_\delta)$

Assuming Lipschitz condition (ex. linear case) over s, it follows that

$$|s - s_\delta| \leq c\|u - u_\delta\|_X$$

Thus any information over the error in the energy norm will allow to get verification.

Actually it is well known that the convergence of s_δ towards s most often goes faster

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$$\begin{aligned} s_\delta - s &= \ell(u_\delta) - \ell(u) \\ &= a(u, \psi) - a(u_\delta, \psi) \\ &= a(u, \psi - \phi_\delta) - a(u_\delta, \psi - \phi_\delta), \quad \forall \phi_\delta \in X_\delta \\ &= a(u - u_\delta, \psi - \phi_\delta), \quad \forall \phi_\delta \in X_\delta \\ &\leq c\|u - u_\delta\|_X \|\psi - \phi_\delta\|_X, \quad \forall \phi_\delta \in X_\delta \end{aligned} \tag{12}$$

Error Estimates

a priori convergence

The convergence of s_δ towards s most often goes faster *ctn.....*

$$s_\delta - s \quad \leq c\|u - u_\delta\|_X \|\psi - \phi_\delta\|_X, \quad \forall \phi_\delta \in X_\delta \tag{13}$$

so that the best fit of ψ in X_δ can be chosen to improve the first error bound that was proposed for $|s - s_\delta|$.
For instance if ψ_δ is the solution of the Galerkin approximation to ψ, we get

$$|s - s_\delta| \leq c\|u - u_\delta\|_X \|\psi - \psi_\delta\|_X$$

But this is just *a priori* business .....

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The basic error reconstruction

In order to get *a posteriori* information, between $\ell(u_h)$ and $\ell(u_\delta)$, we have to get a hand on the residuals in the resolution (in X_δ) of the primal and dual problems. We introduce for any $v \in X$,

$$\mathcal{R}^{pr}(v;\mu) = a(u_\delta, v;\mu) - < f, v >, \quad \mathcal{R}^{du}(v;\mu) = a(v, \psi_\delta;\mu) + \ell(v).$$

Let us now compute the reconstructed errors associated to the previous residuals. These are the solutions to the following problems

$$2\alpha \int \nabla \hat{e}_h{}^{pr(du)} \nabla v_h = \mathcal{R}^{pr(du)}(v_h;\mu), \quad \forall v_h$$

we then have

Theorem Let $s^- = s_\delta - \alpha \int \nabla(\hat{e}_h{}^{pr} + \hat{e}_h{}^{pr})^2$ then $s^- \leq s_h$.

$$s_h - s^- \equiv |s_h - s_\delta|$$

How to transform to a do-able method ??
Black Box again

Outline

Selection of the parameters of the reduced basis

One possibility is to use a random approach.... and hope for the best....
.... generally it works not so badly.
But a more intelligent way can be proposed....
... based on a greedy process that combines the reduced approximation and the error evaluation.

- take a first parameter (randomly)
- use a (one dimensional) reduced basis approach over a set of parameter values (chosen randomly) and select, as a second parameter, the one for which the associated error is the largest.
  this gives now a 2 dimensional reduced basis method.
- use this (2 dimensional) reduced basis approach over the same set of parameters and select, as a third parameter, the one for which the associated error is the largest.
  this gives a 3 dimensional reduced basis method...
- and proceed...

Rational in front of a new problem

When you will come back home.... does your problem fits with RB approximations....

.... try step by step.

- check that your problem is parameter dependant
- compute 100 solutions with your favorite method !!!
- perform a PCA and check the eigenvalue decrease
- perform a Galerkin approximation
- try to find good parameters through projection and perform a Galerkin approximation
- and proceed...

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Close the circle

- The Black Box Approach allows real time solution procedure
- Valid only for linear problem
- at least as it is.
- The use of magic points allows to tackle non linear problem
- Real life problems are non linear... with non constant coefficients

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Nonlinear approximation for high-dimensional partial differential equations

$$-\Delta u(x,y) = f(x,y)$$

with homogeneous Dirichlet boundary conditions

Solution

$$g = \text{argmin}_{u \in H_0^1(\Omega)} \frac{1}{2} \int |\nabla u|^2 - \int fu$$

$$\mathcal{E}(u) = \frac{1}{2} \int |\nabla u|^2 - \int fu = \frac{1}{2} \int |\nabla (u-g)|^2 - \frac{1}{2} \int |\nabla g|^2$$

$$u = \sum_{n=1}^{\infty} r_n(x) s_n(y)$$

Extension to high dimension straightforward

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That's all folks

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